
II. *The Doctrine of Combinations and Alternations, Improv'd and Compleated, by Major* Edward Thornycroft.

IN order to underſtand what follows, it muſt be obſerved,

1ſt, That as in the notation of Powers, a a a a b b b c c is deſign'd by a^4 b^3 c^2, and univerſally p times the poſition of a, q times the poſition of b, r times the poſition of c, by a^p b^p c^r, ſo in things expos'd likewiſe: (unleſs where 'tis propos'd they ſhould be all different) which Indices, as they have here no relation to Powers, but expreſs only the Occurrences of thoſe things to which they reſpectively belong; I therefore call Indices of Occurrences.

2dly, That as often as I ſhall hereafter mention the Combinations or Alternations of the p^s q^s r^s or s^s, (which conſider'd by themſelves are capable of no variation) I mean of thoſe things whoſe Indices they are.

3dly, That m is generally put for the whole number of things expos'd, whether all different or not, *i. e.* equal to the ſum of their Indices; and n, for ſuch a number of them, as each Combination and Alternation muſt conſiſt of; (unleſs preſuppoſed equal) which explains what is hereafter meant by the Combinations and Alternations of m things taken n and n; or of m things taken m and m; and the like Expreſſion, by whatever Symbols the number of things out of which the Combinations and Alternations are to be made, or of which they are to conſiſt, may be deſign'd.

Lemma 1ſt.

If in a right Line, at any diſtances, be plac'd any number of things, a b c d, *&c.* the number of the Intervals a b, b c, c d, *&c.* terminated each by two adjacent things, is one leſs than the number of things.

For, whereas every Interval is terminated by two adjacent things, if to any number of things, be added one thing more, one Interval only is thereby added. Q. E. D.

Lemma 2d.

The number of the Alternations of m things a b c d, *&c.* different each from other, taken m and m, is m times the number of the Alternations of $m-1$ things a b c, taken $m-1$ and $m-1$.

For, (by *Lem.* 1ſt) the laſt Letter d, beſides the poſition it hath, may have $m-2$ poſitions, *viz.* in the Intervals which are between $m-1$ things a b c; but it may alſo have one more, for it may be put firſt of all, it may therefore have m poſitions; and thoſe in all the different Orders, whereof $m-1$ things are capable; which being all the poſſible poſitions of d, in all the varieties of a b c, is all the variety whereof the whole number of things expoſed a b c d, *&c.* is capable. Q. E. D.

Lemma 3d.

The number of the Alternations of m things a b c d, *&c.* different each from other, taken m and m, is equal to $m \times \overline{m-1} \times m\,\overline{m-2} \times \overline{m-3} \times \overline{m-4}$, *&c.* continu'd to m places.

For let $m\,O$, expreſs the number of the Alternations of m things different each from other; $\overline{m-1}\,O$, of $m-1$ things and the like.

'Tis evident that if $m = 1$, It will be $m\,O = m$; for there can be but one order of one thing.

And if m be greater than unity, then will it be (by *Lem.* 2) $m\,O = m \times \overline{m-1}\,O = m \times \overline{m-1} \times \overline{m-2}\,O = m \times \overline{m-1} \times$

$\overline{m-2}$

$\overline{m-2}\times\overline{m-3}$ O =, *&c.* till we have an Equation conſiſting of m places; *i. e.* $= m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}\times$ *&c.* continu'd to m places. Q. E D.

Lemma 4th.

If $m\,\omega$ Expreſs the number of the Alternations of m things $a^p\, b^p\, c^p\, d^q\, e^q\, f^r$ *&c.* taken m and m, and α the number of p's, β the number of q's, γ the number of r's, it will be $m\,\omega = \dfrac{m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}\times\overline{m-4}\times\overline{m-5}\times\&c.\ \text{contiu'd to m places.}}{\overline{p\times\overline{p-1}\times\overline{p-2}\times\&c.}^{\alpha}\times\overline{q\times\overline{q-1}\times\&c.}^{\beta}\times\overline{r\times\overline{r-1}\times\&c.}^{\gamma}}$ each Series continued to p, q, r, *&c.* places reſpectively.

For the number of the Alternations of any number of things, however divided into parts, is produc'd by a continual Multiplication of the Alternations of thoſe things amongſt themſelves reſpectively, which compoſe each part, into the number of their Alternations one amongſt the other; *i. e.* in the preſent caſe (the ſeveral occurrences being ſuppoſed to compoſe the ſeveral parts, and conſequently the number of the Alternations of the things compoſing each part equal to unity) $m\,\omega$ = to the number of the Alternations of the things compoſing the parts one amongſt the other; but the number of their Alternations one amongſt the other, is the ſame in this caſe, as if the things expos'd, being all different, were divided into the ſame parts; for the things which compoſe each part in both caſes, are different from the reſt of the things expos'd; *i. e.* by *Lem.* 3d.

$$m\,\omega = \frac{m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}\times\overline{m-4}\times\overline{m-5}\times\&c.\ \text{continued to m places}}{\overline{p\times\overline{p-1}\times\overline{p-2}\times\&c.}\Big|^{\alpha}\times\overline{q\times\overline{q-1}\times\&c.}\Big|^{\beta}\times\overline{r\times\overline{r-1}\times\&c.}\Big|^{\gamma}}$$ each Series continued to p, q, r places reſpectively. Q. E. D.

Lemma 5th.

The number of the Combinations of m things a b c d, *&c.* different each from other, taken n and n, is equal to $\dfrac{m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}\times\&c.}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\&c.}$ each Series continued to n places

For if the things expos'd be divided in two parts, *viz.* in the *ratio* of n and m—n, 'tis evident that their different binations taken n and n, are produc'd by the Alternations of the things compoſing the parts one amongſt the other: And therefore the number of thoſe = to the number of theſe = to the number of the Alternations of m things taken m and m, the Indices of whoſe occurrences are n and $m-n = \frac{m \times m-1 \times m-2 \times m-3 \times \&c. \text{ continued to m places}}{n \times n-1 \times \&c. \times m-n \times m-n-1 \times \&c. \text{ each Series continued to n and m-n}}$ places reſpectively (by *Lem.* 4th) *i. e.* becauſe $n + m-n = m = \frac{m \times m-1 \times m-2 \times m-3 \&c.}{n \times n-1 \times n-2 \times n-3 \&c.}$ each Series continued to n places Q. E. D.

But the number of the Alternations in every Combination is $= n \times n-1 \times n-2 \times n-3 \times$, *&c.* continued to n places, by *Lem.* 3d therefore.

Lemma 6th.

The number of the Alternations of m things a b c d, *&c.* different each from other, taken n and n, is $= m \times m-1 \times m-2 \times m-3 \times$, *&c.* continued to n places. Q. E. D.

Scholium.

Since in the things expos'd the ſame things may occur more than once, and alſo n be leſs than m, the Indices of the occurrences which are in ſome of the Combinations of m things taken n and n, may differ from thoſe which are in others; but thoſe Combinations, the Indices of whoſe occurrences are the ſame, are ſaid to be in the ſame form: Therefore whereas n is equal to the ſum of the Indices which are in each Combination taken n and n, if n be expreſs'd by all the different Combinations of ſuch Indices only (being integer numbers) whereof no one may exceed the higheſt Index of the things expos'd, and being more than one in a Combination, are each of them, which are in the ſame Combination, comprehended in a diſtinct Index thereof; theſe Expreſſions of n will neceſſarily be the ſeveral forms of the Combinations taken n and n, whereof m things are capable: Whence is deriv'd a General

The.

Theorem for finding the Combinations and Alternations of m things taked n and n univerſally: *i. e.* Whether m conſiſt of things all different or not, and whether n be equal to, or leſs than m.

Theorem.

If n be expreſs'd, according to all the different Forms of Combination which the things expos'd are capable of,

$$\text{and}\left\{\begin{array}{l} p = \text{the higheſt Index} \\ q = \text{the next higheſt} \\ r = \text{the next higheſt} \\ \text{ſ} = \text{the next higheſt} \\ \&c. \end{array}\right. \left\{\begin{array}{l} \alpha = \text{the number of p's} \\ \beta = \text{the number of q's} \\ \gamma = \text{the number of r's} \\ \delta = \text{the number of ſ's} \end{array}\right\} \begin{array}{l} \text{In every} \\ \text{form of} \\ \text{Combi-} \\ \text{nation.} \end{array}$$

$$\left\{\begin{array}{l} A = \text{the number of all Indices not leſs than p} \\ B = \text{the number of all Indices not leſs than q} \\ C = \text{the number of all Indices not leſs than r} \\ D = \text{the number of all Indices not leſs than ſ} \\ \&c. \end{array}\right\} \begin{array}{l} \text{Which are} \\ \text{in the} \\ \text{things} \\ \text{Expos'd} \end{array}$$

and $b = \alpha + \beta$, $c = b + \gamma$, $d = c + \delta$, *&c.*

I ſay the number of the Combinations of m things taken n and n, in any one Form of Combination, ſhall be

$$\frac{A \times A-1 \times A-2}{\alpha \times \alpha-1 \times \alpha-2} \&c. \times \frac{B-\alpha \times B-\alpha-1}{\beta \times \beta-1} \times \&c. \times \frac{C-b \times C-b-1}{\gamma \times \gamma-1} \times \&c. \times \frac{D-c \times D-c-1}{\delta \times \delta-1}$$

&c. continued to ſo many Terms as there are different Indices in the form of Combination and each Term to α, β, γ, δ, *&c.* places reſpectively, and this number multiply'd into

$$\frac{\times\, n-1 \times n-2 \times n-3 \times n-4 \times n-4 \times n-5 \times n-6 \text{ \&c. continued to n places,}}{\overline{p \times p-1 + p-2 \times \&c.}|^{\alpha} \times \overline{q \times q-1 \times \&c}|^{\beta} \times \overline{r \times r-1 \times \&c.}|^{\gamma} \times \text{\&c. each Series continued to p, q, r,}}$$

&c. places reſpectively, ſhall be the number of their Alternations.

But the ſum of all the Combinations and Alternations which are in every Form of n, ſhall be the whole number of Combinations and Alternations of m things taken n and n.

Demonſtration.

Firſt, Then 'tis evident, that thoſe Combinations, which are in different forms, differ from each other.

Again, 'Tis evident that the Combinations of m things, as $a^p\ b^p\ c^p\ d^p\ e^q\ f^q\ g^q\ h^r\ i^r$, *&c.* (the Indices ſimply conſider'd) taken n and n, in a form wherein are p^s q^s and r^s, ſhall be equal to the number of the Combinations of the p^s, which are in the things expos'd, taken α and α, multiply'd into the number of the Combinations of the q^s taken β and β multiply'd into the number of the Combinations of the r^s taken γ and γ.

But becauſe p and all leſſer Indices are comprehended in every Index which is greater than themſelves; therefore is A = to the number of p^s which are in the things expos'd, and for the ſame reaſon, would B = the number of the q^s, and C the number of r^s: But the number of the p^s, which are in every form of Combination, is $= \alpha$; therefore is $B - \alpha =$ to the number of q^s; alſo becauſe the number of p^s and q^s together, which are in every form of Combination, wherein there are q^s, is $= \alpha + \beta = b$; therefore is $C - b =$ to the number of r^s, and ſo on, how many ſoever were the different Indices in any form of Combination.

But (by *Lem.* 5th) the number of the Combinations of the p^s, which are in the things expos'd, whoſe number is A, taken α and α, is $= \frac{A \times A - 1 \times A - 2}{\alpha \times \alpha - 1 \times \alpha - 2}$ *&c.* continu'd to α places, and the number of the Combinations of the q^s, whoſe number is $B - \alpha$, taken β and β, is $= \frac{B - \alpha \times B - \alpha - 1}{\beta \times \beta - 1}$ $\frac{\times B - \alpha - 2}{\times \beta - 2}$ *&c.* continued to β places, and the number of the Combinations of the r^s, whoſe number is $C - b$, taken γ and γ, is $= \frac{C - b \times C - b - 1}{\gamma \times \gamma - 1}$ *&c.* continued to γ places Q. E. D.

But

But every Combination, in one and the ſame form, affords the ſame number of Alternations: Therefore the number of Alternations, in any one form, is ſo many times the number of Combinations, as is the number of Alternations in any one of theſe Combinations.

But (by *Lem.* 4th) the number of Alternations in any of thoſe Combinations ſhall be

$$\frac{n\times n-1\times n-2\times n-3\times n-4+n-5\times n-6\times \&c.\ \text{continued to n places}}{\overline{p\times p-1\times p-2\times \&c.}|^{\alpha}\times\overline{q\times q-2\times \&c.}|^{\beta}\times\overline{r\times r-1\times \&c.}|^{\gamma}\times\ \text{each Series continued to p q r \&c.}}$$

places reſpectively. Q. E. D.

Now to make an application of this general Rule to thoſe particular caſes which have already been conſider'd by others, and which are contain'd in our 3d, 4th, 5th and 6th *Lemma's*, and by us more generally demonſtrated; I ſay

If n = m, there can be but one form of Combination, and but one Combination in that form; and therefore the number of Alternations $=\frac{m\times m-1\times m-2\times m-3\times m-4\times \&c.\ \text{contin'd to m places}}{\overline{p\times p-1\times \&c.}|^{\alpha}\times\overline{q\times q-1\times \&c.}|^{\beta}\times\overline{r\times \&c}|^{\gamma}\times}$ *&c.* each Series to p q r, *&c.* places reſpectively, *i. e.* (if p = 1) = m × m−1 × m−2 × m−3 × m−4 × *&c.* continu'd to m places, which are the caſes of the 4th and 3d *Lemma's*.

But if the things expos'd are all different, and n be leſs than m, which is the caſe of the 5th and 6th *Lemma's*, then alſo can there be but one form of Combination, and it will be A = m & α = n, and the whole number of Combinations $=\frac{A\times A-1\times A-2\times \&c.}{\alpha\times\alpha-1\times\alpha-2\times\&c.}$ *i. e.* $=\frac{m\times m-1\times m-2\times\&c}{n\times n-1\times n-2\times\&c}$ each Series continued to n places, and therefore the number of Alternations = m × m−1 × m−2 × *&c.* continu'd to n places.

But fully to Illuſtrate this Theorem, which, as deliver'd in general, may ſeem ſomewhat too Abſtracted, to be commonly underſtood, I ſhall ſubjoyn one ſhort Example.

Example.

Let the things expos'd be a a a b b b c c, or according to our way of notation $a^3\,b^3\,c^2$; 'Tis requir'd to find the number of their Combinations and Alternations taken 4 and 4.

Then (becauſe in the things expos'd, there is no one thing occurs more than thrice, nor more than three things different from each other) will all the forms of Combination, which the things expos'd are capable of, be theſe,

$$viz. \left\{\begin{array}{l} 3\,.\,1 \\ 2\,.\,2 \\ 2\,.\,1\,.\,1 \end{array}\right\} \text{Then}$$

In the 1ſt form will p $= 3$, q $= 1$, $\alpha = 1$, $\beta = 1$, A $= 2$, B $= 3$
In the 2d form will p $= 2$, ——, $\alpha = 2$, ——, A $= 3$, ——
In the 3d form will p $= 2$, q $= 1$, $\alpha = 1$, $\beta = 2$, A $= 3$, B $= 3$

The number of Combinations in the 1ſt form $= \frac{A}{\alpha} \times \frac{\overline{B-\alpha}}{\beta} = \frac{2}{1} \times \frac{2}{1} = 4$

The number of Combinations in the ſecond form $= \frac{A \times \overline{A-1}}{\alpha \times \overline{\alpha-1}} = \frac{3 \times 2}{2 \times 1} = 3$

The number of Combinations in the 3d form $= \frac{A}{\alpha} \times \frac{\overline{B-\alpha} \times \overline{B-\alpha-1}}{\beta \times \overline{\beta-1}} = \frac{2 \times 1}{2 \times 1} = 3$

And the whole number of Combinations $= 10$

Alſo the number of Alternations.

In the 1ſt form $= 4 \times \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{\overline{p \times \overline{p-1} \times \overline{p-2}}|^{\alpha} \times q^{\beta}} = 4 \times \frac{4 \times 3 \times 2 \times 1}{\overline{3 \times 2 \times 1}|^{1} \times} = 4 \times 4 = 16$

In the 2d form $= 3 \times \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{\overline{p \times \overline{p-1}}|^{\alpha}} = 4 \times \frac{4 \times 3 \times 2 \times 1}{\overline{2 \times 1}|^{2}} = 3 \times 6 = 18$

In the 3d form $= 3 \times \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{\overline{p \times \overline{p-1}}|^{\alpha} \times q^{\beta}} = 3 \times \frac{4 \times 3 \times 2 \times 1}{\overline{2 \times 1}|^{1} \times 1^{2}} = 3 \times 12 = 36$

And the whole number of Alternations $= 70$

Many are the Properties of this *Theorem*, in common with others, as, To find the *Unciæ* of a Multinomial rais'd raiſ'd to any integer power. To raiſe an Infinite Series to an integer power, though of an interrupted Order, without introducing any thing immaterial, or which muſt afterwards be expung'd; and many others. But then ſo many Terms of the Series muſt be taken in at firſt as ſhall ſerve to the purpoſes of the intended approximation, otherwiſe as often as it ſhall fall ſhort of that, the Operation muſt be began *de novo.*

Many likewiſe are the Properties peculiar to this *Theorem*, and great Variety of Problems might be fram'd; and I ſcruple not to ſay, many may occur in Practice, which are ſolvable by this, and no other Method whatever.

Hence may be found the number of Words whereof the 24 Letters are capable, from one Letter in each Word, to any number of Letters given.

Hence may be found the number of all Numbers, to any given number of Places, which may be produc'd from any number of Figures given.

Hence alſo the Compaſs of a Muſical Inſtrument being given, the Time and number of the Bars, whereof each Tune ſhall conſiſt, the number of Tunes may be found which that Inſtrument is capable of.

To give an inſtance of the prodigious variety that there is in Muſick, I have Calculated the number of Tunes in Common Time, conſiſting of eight Bars each, which may be play'd on an Inſtrument of one Note Compaſs only, and it is this, *viz.* 27584. 270157. 013570. 368586. 999728. 299176. whereas the Changes on 24 Bells is but 620448. 401733. 239439. 360000, which is but $\frac{1}{444588.\ 604583}$ of the number of Tunes, and yet Dr *Wallis* in his *Algebra* demonſtrates, could not be diſpatched in 31557. 600000. 000000 years.

If then the Inſtrument were of as many Notes Compaſs as any Inſtrument now in uſe, how prodigiouſly [illegible]

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number of Tunes be encereas'd; the Calculation of which (tho much more intricate and operoſe) would be equally attainable by our Theorem.

III. *Of Oſſifications or Petrifactions in the Coats of Arteries, particularly in the Valves of the Great Artery, by* William Cowper, *Surgeon, and F. R. S.*

HOw far Anatomical Enquiries inform in the true cauſes of Diſeaſes, which have been aſcribed to the want of Spirits in ſome, and Radical Moiſture in Aged People, *&c.* may be in ſome meaſure ſeen by two Obſervations, among others, publiſht in the Tranſactions No 280: The firſt there mentioned, pag. 1195, is of a young Gentlewoman, in whom the *Parietes*, or Membranes, that compoſe the Trunks of the Arteries of the Arm near the *Axilla*, being very much thickened, ſo that the *Diameter* of its Bore was leſſened to more than a third part of its natural ſize; inſomuch that a part of the Trunk of the Artery cut Tranſverſel very much reſembled a bit of the ſtem of a Tobacco-pipe, its ſides were ſo thick, and its Bore conſequently ſo much leſſened: The other was of the Trunks of the Arteries of the Leg, pag. *ib.* that were Obſtructed by Petrifactions or Oſſifications, in a perſon about the 67th year of his Age. Since which I have met with ſeveral of the like Inſtances in people of years, particularly in the Leg of an old Gentleman, whoſe Toes and Foot were Sphacelated, the Arteries of whoſe Leg I have ſtill by me, and have ſent them herewith Injected, as much as they could be, with Red Wax; in which the Oſſifications diminiſhing their Channels in ſome places, and totally obſtructing them in others,